Nonstandard methods in combinatorial number theory

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AILA 2017

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- Is it true that for every $A \subseteq \mathbb{N}$ if $BD(A) > 0$ then there exist infinite sets X, Y such that $X + Y \subseteq A$?

Terminology

We say that a polynomial $P(x_1, ..., x_n)$ is partition regular on $\mathbb{N} = \{1, 2, ...\}$ if whenever the natural numbers are finitely colored there exists a monochromatic solution to the equation $P(x_1, ..., x_n) = 0$.

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1. *$P(x_1, ..., x_n)$ is partition regular on $\mathbb{N}$;*
2. *there is a nonempty subset J of $\{1, ..., n\}$ such that $\sum_{j \in J} a_j = 0$.*

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Fact: Since the '70s, ultrafilters have been one of the main tools to prove results in combinatorial number theory.

Translation in terms of Ultrafilters

Definition

Let $P(x_1, ..., x_n)$ be a polynomial, and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$. We say that $\mathcal{U}$ is a ***$\sigma_{\mathbf{P}}$-ultrafilter*** *if and only if for every set $A \in \mathcal{U}$ there are $a_1, ..., a_n \in A$ such that $P(a_1, .., a_n) = 0$.*

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$$G_{\mathcal{U}} = \{\alpha \in {}^*\mathbb{N} \mid \mathcal{U} = \mathfrak{U}_\alpha\},$$

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where $\mathfrak{U}_\alpha = \{A \subseteq \mathbb{N} \mid \alpha \in {}^*A\}$.

If we work in an extension that allows for the iteration of the $*$-map, we also have that

$$\mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta = \mathfrak{U}_{{}^*\beta+\alpha}, \mathfrak{U}_\alpha \odot \mathfrak{U}_\beta = \mathfrak{U}_{{}^*\beta\cdot\alpha}.$$

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$$\alpha \cdot \ {}^*\alpha + \beta \cdot \ {}^*\alpha = \gamma \cdot \ {}^*\alpha$$

and we conclude setting $x = \alpha \cdot \ {}^*\alpha, y = \beta \cdot \ {}^*\alpha, u = \alpha, v = \ {}^*\alpha$.

Main Partition Regularity Result/1

Theorem

Let $\mathfrak{F}$ be the family of functions whose PR on $\mathbb{N}$ is witnessed by at least an ultrafilter $\mathcal{U} \in \mathbb{I}(\odot) \cap \overline{K(\odot)} \cap \overline{\mathbb{I}(\oplus)} \cap \mathcal{BD}$.

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Then $\mathfrak{F}$ includes:

- *Every Rado polynomial*

$$c_1x_1 + \ldots + c_nx_n + P(y_1, \ldots, y_k)$$

  with injectivity $|\{x_1, \ldots, x_n\}| \geqslant n-1$ and $|\{y_1, \ldots, y_k\}| = k$, and with injectivity $|\{x_1, x_2\}| = 2$ when $n = 2$ and $k = 1$, and with full injectivity $|\{x_1, \ldots, x_n, y_1, \ldots, y_k\}| = n + k$ when $P \neq 0$ is linear;

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- *Every polynomial of the form*

$$\sum_{i=1}^{n} c_i\, x_i \left(\prod_{j \in F_i} y_j \right)$$

  where $\sum_{i=1}^{n} c_i\, x_i$ is a Rado polynomial and sets $F_i \subseteq \{1, \ldots, m\}$, with full injectivity when $n > 2$, and with injectivity $|\{x_1, x_2, y_1, \ldots, y_m\}| \geqslant m+1$ when $n = 2$;

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- *Every function f of the form*

$$f(x, y_1, \ldots, y_k) = x - \prod_{i=1}^{k} y_i$$

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- *Every function f of the form*

$$f(x, y_1, \ldots, y_k) = x - \prod_{i=1}^{k} y_i^{a_i}$$

with full injectivity $|\{x, y_1, \ldots, y_k\}| = k + 1$*, whenever the exponents* $a_i \in \mathbb{Z}$ *satisfy* $\sum_{i=1}^{n} a_i = 1$*.*

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1. The polynomial $P(x, y) = x^3 + 2x + y^3 - 2y$ is not PR.
2. If $k \notin \{n, m\}$, the polynomial $x^n + y^m - z^k$ is not PR.

Thank You!

email: lorenzo.luperi.baglini@univie.ac.at